Partial Derivatives

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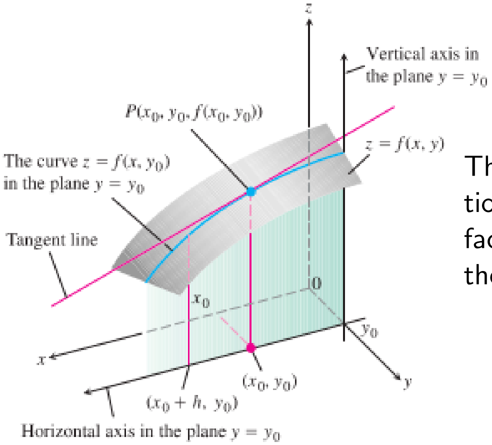
Overview

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a partial derivative.

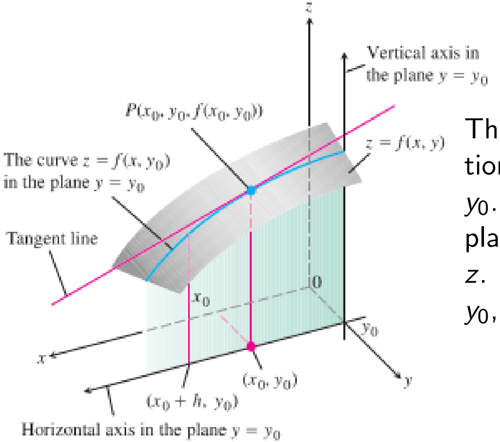
In this lecture, we see how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the rules for differentiating functions of a single variable.

The idea of differentiability for functions of several variables requires more than the existence of the partial derivatives, but we will see that differentiable functions of several variables behave in the same way as differentiable single-variable functions.

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$.



The picture shows that the intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.



This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x; the vertical coordinate is z. The y-value is held constant at y_0, so y is not a variable.

Partial Derivatives of a Function of Two Variables

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used. The partial derivative symbol is variously pronounced as "del", "partial dee", "**doh**", or "**dabba**".

In the definition, h represents a real number, positive or negative.

Definition 1.

The partial derivative of f(x,y) with respect to x *at the point* (x_0, y_0) *is*

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Partial Derivatives of a Function of Two Variables

An equivalent expression for the partial derivative is

$$\frac{d}{dx} f(x, y_0)\bigg|_{x=x_0}.$$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0).

The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f / \partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.

Partial Derivatives of a Function of Two Variables

We use several notations for the partial derivative:

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0), \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}, \quad \text{and} \quad f_x, \frac{\partial f}{\partial x}, z_x, \text{ or } \frac{\partial z}{\partial x}.$$

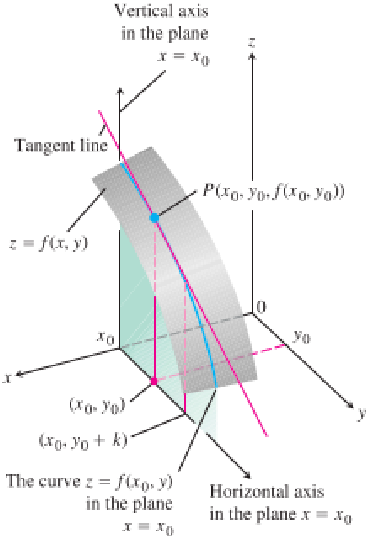
The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0.

Definition 2.

The partial derivative of f(x,y) with respect to y *at the point* (x_0, y_0) *is*

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.



The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

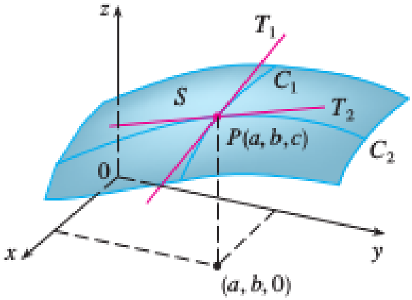
Partial Derivatives of a Function of Two Variables

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ is the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0.

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x:

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$

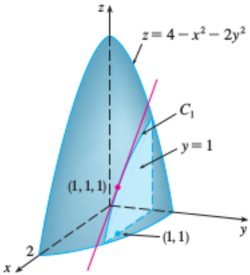
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

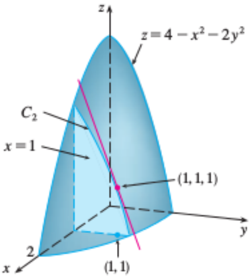


Partial Derivatives of a Function of Two Variables

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$.

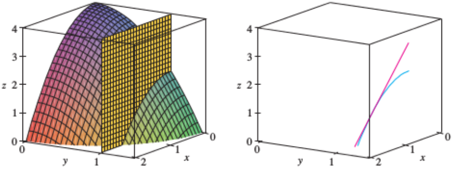
Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2, x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$.



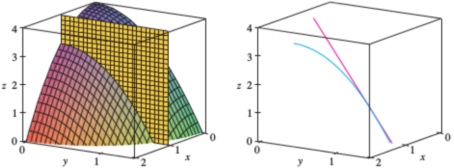


Partial Derivatives of a Function of Two Variables

The following figures are computer-drawn counterparts to the last figures. Part (a) shows the plane $y = 1$ intersecting the surface to form the curve C_i and part (b) shows the curve C_i and the tangent line T_i, $i = 1, 2$.

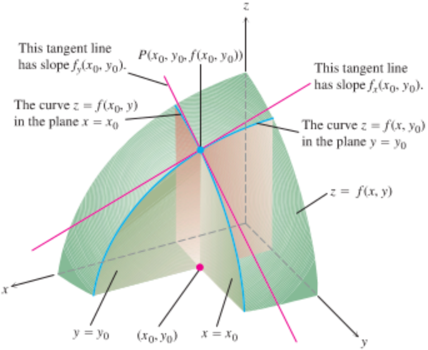


(a) (b)



Partial Derivatives of a Function of Two Variables

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$.



The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Is the plane they determine tangent to the surface at P? We will see that it is for the *differentiable* functions, we will learn how to find the tangent plane. First we have to learn more about partial derivatives themselves.

Calculations

The definitions of $\partial f/\partial x$ and $\partial f/\partial y$ give us two different ways of differentiating f at a point: with respect to x in the usual way while treating y as a constant and with respect to y in the usual way while treating x as a constant. As the following examples show, the values of these partial derivatives are usually different at a given point (x_0, y_0).

Example 3.

Find the values of $\partial f/\partial x$ *and* $\partial f/\partial y$ *at the point* $(4, -5)$ *if*

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution : To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

Solution (contd...)

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

Example 4.

Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution : We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Example 5.

Find f_x and f_y as functions if $f(x,y) = \frac{2y}{y+\cos x}$.

Solution : We treat f as a quotient. With y held constant, we get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$
$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}.$$

With x held constant, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$
$$= \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}.$$

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

Example 6.

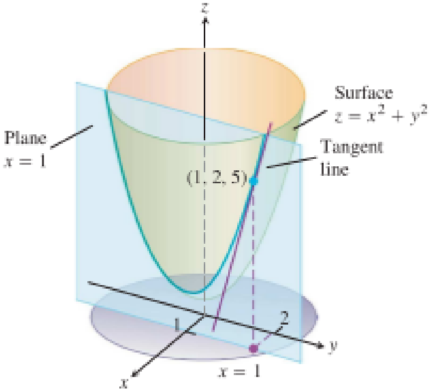
Find $\partial z/\partial x$ if the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exists.

We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\begin{aligned} \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} \\ \left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} &= 1 \\ \frac{\partial z}{\partial x} &= \frac{z}{yz - 1}. \end{aligned}$$

Example 7.

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.



The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point $(1, 2, 5)$.

Solution

The slope is the value of the partial derivative $\partial z/\partial y$ at $(1,2)$:

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2+y^2)\right|_{(1,2)} = 2y|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1+y^2)\right|_{y=2} = 2y|_{y=2} = 4.$$

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example 8.

If x, y, and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

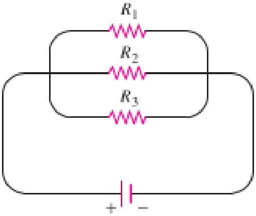
$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) \\ &= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) = 3x \cos(y + 3z).\end{aligned}$$

Example 9.

If resistors of R_1,R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Find the value of $\partial R/\partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.



Resistors arranged this way are said to be connected in parallel. Each resistor lets a portion of the current through. Their equivalent resistance R is calculated with the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Solution

To find $\partial R/\partial R_2$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2} = \left(\frac{R}{R_2}\right)^2.$$

Solution (contd...)

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3+2+1}{90} = \frac{6}{90} = \frac{1}{15},$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \left(\frac{1}{9}\right).$$

Thus at the given values, a small change in the resistance R_2 leads to a change in R about $1/9$th as large.

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there.

This is different from functions of a single variable, where the existence of a derivative implies continuity.

We shall see that if the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0), however, then f is continuous at (x_0, y_0).

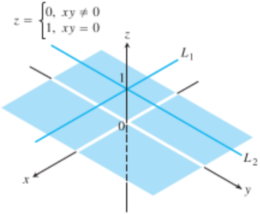
Example 10.

Let $f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0. \end{cases}$

(a) *Find the limit of* f *as* (x, y) *approaches* $(0, 0)$ *along the line* $y = x$.

(b) *Prove that* f *is not continuous at the origin.*

(c) *Show that both partial derivatives* $\partial f/\partial x$ *and* $\partial f/\partial y$ *exist at the origin.*



The graph of $f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$ consists of the linesL_1 and L_2 and the four open quadrants of the xy-plane. The function has partial derivatives at the origin but is not continuous there.

Solution

(a) Since $f(x,y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x,y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(b) Since $f(0,0) = 1$, the limit in part (a) proves that f is not continuous at $(0,0)$.

(c) To find $\partial f/\partial x$ at $(0,0)$, we hold y fixed at $y = 0$. Then $f(x,y) = 1$ for all x, and the graph of f is the line L_1 in the figure. The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0,0)$. Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y, so $\partial f/\partial y = 0$ at $(0,0)$.

Relation Between Differentiability and Continuity

The above example suggests that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives.

We shall define differentiability for functions of two variables (which is slightly more complicated than for single-variable functions) and then revisit the connection to continuity.

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}, \quad \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on.

Notice the order in which the mixed partial derivatives are taken:

$\frac{\partial^2 f}{\partial x \partial y}$	Differentiate first with respect to y, then with respect to x.
$f_{yx} = (f_y)_x$	Means the same thing.

Example 11.

If $f(x,y) = x\cos y + ye^x$, *find the second-order derivatives*

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution : The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$

The Mixed Derivative Theorem

We have noticed in the above example that the mixed second-order partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

are equal. This is not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous, as stated in the following theorem.

Theorem 12 (The Mixed Derivative Theorem).

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Clairaut's Theorem

The Mixed Derivative Theorem is also known as **Clairaut's Theorem**, named after the French mathematician Alexis Clairaut who discovered it.

The Mixed Derivative Theorem says that to calculate a mixed second-order derivative, we may differentiate in either order, provided the continuity conditions are satisfied. This ability to proceed in different order sometimes simplifies our calculations.

Example 13.

Find $\partial^2 w/\partial x \partial y$ *if*

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution

The symbol $\partial^2 w/\partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x \partial y = 1$ as well. We can differentiate in either order because the conditions of "The Mixed Derivative Theorem" hold for w at all points (x_0, y_0).

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx},$$
$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

Example 14.

Find f_{yxyz} *if* $f(x,y,z) = 1 - 2xy^2z + x^2y$.

Solution : We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

Differentiability

The starting point for differentiability is not the difference quotient we saw in studying single-variable functions, but rather the idea of increment. Recall that if $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$$

in which $\varepsilon \to 0$ as $\Delta x \to 0$.

For functions of two variables, the analogous property becomes the definition of differentiability.

Theorem 15 (The Increment Theorem for Functions of Two Variables).

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

Definition 16.

A function $z = f(x, y)$ *is* differentiable at (x_0, y_0) *if* $f_x(x_0, y_0)$ *and* $f_y(x_0, y_0)$ *exist and* Δz *satisfies an equation of the form*

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ *as both* $\Delta x, \Delta y \to 0$. *We call* f differentiable *if it is differentiable at every point in its domain, and say that its graph is a* smooth surface.

Because of this definition, an immediate corollary of "Increment Theorem for Functions of Two Variables" is that a function is differentiable at (x_0, y_0) if its first partial derivatives are *continuous* there.

Corollary 17.

If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

If $z = f(x, y)$ is differentiable, then the definition of differentiability assures that $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ approaches 0 as Δx and Δy approach 0. This tells us that a function of two variables is continuous at every point where it is differentiable.

Theorem 18 (Differentiability Implies Continuity).

If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Remark

A function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0).

Remember, however, that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist.

Existence alone of the partial derivatives at that point is not enough, but continuity of the partial derivatives guarantees differentiability.

Recall

The Chain Rule for functions of a single variable says that when

- $w = f(x)$ is a differentiable function of x, and
- $x = g(t)$ is a differentiable function of t,

then

- w is a differentiable function of t, and
- dw/dt can be calculated by the formula $\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$.

For functions of two or more variables, the Chain Rule has several forms. The form depends on how many variables are involved, but once this is taken into account, it works like the Chain Rule for functions of a single variable.

The Chain Rule formula for a differentiable function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

Theorem 19 (Chain Rule for Functions of Two Independent Variables).

If $w = f(x, y)$ is differentiable and if $x = x(t)$,$y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

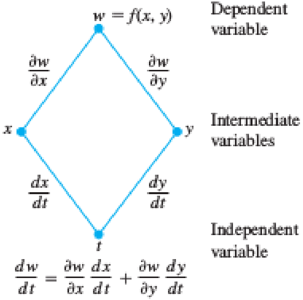
Branch Diagram : Functions of Two Variables

Often we write $\partial w/\partial x$ for the partial derivative $\partial f/\partial x$, so we can rewrite the Chain Rule in the form

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

To remember the Chain Rule picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



We can now probably predict the Chain Rule for functions of three variables, as it only involves adding the expected third term to the two-variable formula.

Theorem 20 (Chain Rule for Functions of Three Independent Variables).

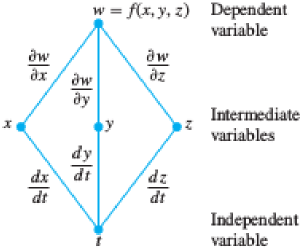
If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Branch Diagram : Functions of Two Variables

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



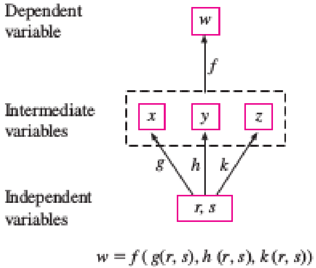
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Functions Defined on Surfaces

If we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on the earth's surface, we might prefer to think of x, y, and z as functions of the variables r and s that give the point's longitudes and latitudes.

If $x = g(r, s), y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of r and s with the composite function

$$w = f(g(r, s), h(r, s), k(r, s)).$$



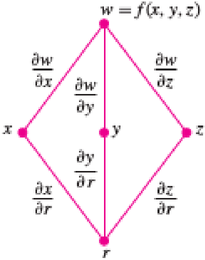
Under the conditions stated below, w has partial derivatives with respect to both r and s that can be calculated in the following way.

Theorem 21 (Chain Rule for Two Independent Variables and Three Intermediate Variables).

Suppose that $w = f(x, y, z), x = g(r, s), y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

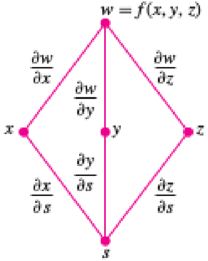
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Branch Diagram for $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Branch Diagram for $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

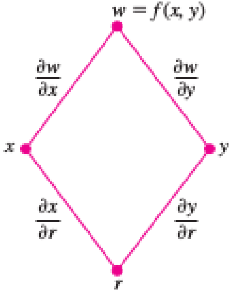
Branch Diagram for $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$

If f is a function of two variables instead of three, each equation in the above Theorem becomes correspondingly one term shorter.
If $w = f(x, y), x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

Branch Diagram

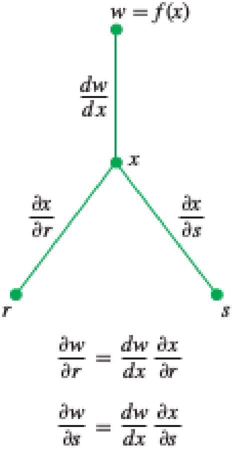
If f is a function of x alone, our equations are even simpler.

If $w = f(x, y), x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$



Theorem 22 (A Formula for Implicit Differentiation).

Suppose that $F(x, y)$ *is differentiable and that the equation* $F(x, y) = 0$ *defines* y *as a differentiable function of* x*. Then at any point where* $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Functions Defined on Surfaces

The result in Theorem 22 is easily extended to three variables.

Suppose that the equation $F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$. Then for all all (x, y) in the domain of f, we have

$$F(x, y, f(x, y)) = 0.$$

Assuming that F and f are differentiable functions, we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to the independent variables x and y. Hence we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \tag{1}$$

Implicit Function Theorem

An important result from advanced calculus, called the **Implicit Function Theorem**, states the conditions for which our results in Equations (1) are valid.

If the partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0), and if for some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by Equations (1).

We have seen several different forms of the Chain Rule in this section, but each one is just a special case of one general formula. When solving particular problems, it may help to draw the appropriate branch diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom.

To find the derivative of the dependent variable with respect to the selected independent variable, start at the dependent variable and read down each route of the branch diagram to the independent variable, calculating and multiplying the derivatives along each route. Then add the products found for the different routes.

In general, suppose that $w = f(x, y, \ldots, v)$ is a differentiable function of the variables $x, y, \ldots, v$ (a finite set) and the $x, y, \ldots, v$ are differentiable functions of $p, q, \ldots, t$ (another finite set).

Then w is a differentiable function of the variables p through t, and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \ldots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

The other equations are obtained by replacing p by $q, \cdots, t$, one at a time.

One way to remember this equation is to think of the right-hand side as the dot product of two vectors with components

$$\overbrace{\underbrace{x}_{\text{real}} + \underbrace{iy}_{\text{imaginary}}}^{\text{complex number}}$$

$$\underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \ldots, \frac{\partial w}{\partial v}\right)}_{\substack{\text{Derivatives of } w \text{ with} \\ \text{respect to the} \\ \text{intermediate variables}}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \ldots, \frac{\partial v}{\partial p}\right)}_{\substack{\text{Derivatives of the} \\ \text{intermediate variables with} \\ \text{respect to the selected} \\ \text{independent variable}}} \quad .$$

Exercise 23.

Find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x,y) = \tan^{-1}(y/x)$
2. $f(x,y) = (x^3 + (y/2))^{2/3}$
3. $f(x,y) = (x+y)/(xy-1)$
4. $f(x,y) = e^{-x}\sin(x+y)$
5. $f(x,y) = \sin^2(x-3y)$
6. $f(x,y) = \displaystyle\int_x^y g(t)dt \quad (g \text{ continuous for all } t)$
7. $f(x,y) = \displaystyle\sum_{n=0}^{\infty}(xy)^n \; (|xy| < 1)$

Solution for (1),(2),(3) and (4) in Exercise 23

1. $\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2[1+(\frac{y}{x})^2]} = -\frac{y}{x^2+y^2}$,
   $\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x[1+(\frac{y}{x})^2]} = \frac{x}{x^2+y^2}$
2. $\frac{\partial f}{\partial x} = \frac{2x^2}{\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$, $\frac{\partial f}{\partial y} = \frac{1}{3\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$
3. $\frac{\partial f}{\partial x} = \frac{(xy-1)(1)-(x+y)(y)}{(xy-1)^2} = \frac{-y^2-1}{(xy-1)^2}$, $\frac{\partial f}{\partial y} = \frac{(xy-1)(1)-(x+y)(x)}{(xy-1)^2} = \frac{-x^2-1}{(xy-1)^2}$
4. $\frac{\partial f}{\partial x} = -e^{-x}\sin(x+y) + e^{-x}\cos(x+y)$, $\frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$

Solution for (5),(6) and (7) in Exercise 23

5. $\frac{\partial f}{\partial x} = 2\sin(x-3y)\cdot\frac{\partial}{\partial x}\sin(x-3y) =$
   $2\sin(x-3y)\cos(x-3y)\cdot\frac{\partial}{\partial x}(x-3y) = 2\sin(x-3y)\cos(x-3y)$,
   $\frac{\partial f}{\partial y} = 2\sin(x-3y)\cdot\frac{\partial}{\partial y}\sin(x-3y) =$
   $2\sin(x-3y)\cos(x-3y)\cdot\frac{\partial}{\partial y}(x-3y) = -6\sin(x-3y)\cos(x-3y)$
6. $\frac{\partial f}{\partial x} = -g(x)$, $\frac{\partial f}{\partial y} = g(y)$
7. $f(x,y) = \sum_{n=0}^{\infty}(xy)^n$,
   $|xy| < 1 \Rightarrow f(x,y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2}\cdot\frac{\partial}{\partial x}(1-xy) = \frac{y}{(1-xy)^2}$
   and $\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2}\cdot\frac{\partial}{\partial y}(1-xy) = \frac{x}{(1-xy)^2}$

Exercise 24.

Find f_x, f_y and f_z.

1. $f(x, y, z) = 1 + xy^2 - 2z^2$
2. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
3. $f(x, y, z) = \sin^{-1}(xyz)$
4. $f(x, y, z) = \ln(x + 2y + 3z)$
5. $f(x, y, z) = \tanh(x + 2y + 3z)$
6. $f(x, y, z) = \sinh(xy - z^2)$

Solution for Exercise 24

1. $f_x = y^2,\ f_y = 2xy,\ f_z = -4z$
2. $f_x = -x(x^2 + y^2 + z^2)^{-3/2},\ f_y = -y(x^2 + y^2 + z^2)^{-3/2},$
   $f_z = -z(x^2 + y^2 + z^2)^{-3/2}$
3. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}},\ f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}},\ f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$
4. $f_x = \frac{1}{x+2y+3z},\ f_y = \frac{2}{x+2y+3z},\ f_z = \frac{3}{x+2y+3z}$
5. $f_x = sech^2(x + 2y + 3z),\ f_y = 2\,sech^2(x + 2y + 3z),$
   $f_z = 3\,sech^2(x + 2y + 3z)$
6. $f_x = y\,cosh(xy - z^2),\ f_y = x\,cosh(xy - z^2),\ f_z = -2z\,cosh(xy - z^2)$

Exercise 25.

Find the partial derivative of the function with respect to each variable.

1. $f(t, \alpha) = \cos(2\pi t - \alpha)$
2. $g(r, \theta, z) = r(1 - \cos\theta) - z$
3. Work done by the heart

$$W(P, V, \delta, v, g) = PV + \frac{V\delta v^2}{2g}$$

4. Wilson lot size formula

$$A(c, h, k, m, q) = \frac{km}{q} + cm + \frac{hq}{2}$$

Solution for Exercise 25

1. $\frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha)$, $\frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$
2. $\frac{\partial g}{\partial r} = 1 - \cos\theta$, $\frac{\partial g}{\partial \theta} = r\sin\theta$, $\frac{\partial g}{\partial z} = -1$
3. $W_p = V$, $W_v = P + \frac{\delta v^2}{2g}$, $W_\delta = \frac{Vv^2}{2g}$, $W_v = \frac{2V\delta v}{2g} = \frac{V\partial v}{g}$, $W_g = -\frac{V\delta v^2}{2g^2}$
4. $\frac{\partial A}{\partial c} = m$, $\frac{\partial A}{\partial h} = \frac{q}{2}$, $\frac{\partial A}{\partial k} = \frac{m}{q}$, $\frac{\partial A}{\partial m} = \frac{k}{q} + c$, $\frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$

Exercise 26.

Find all the second-order partial derivatives of the functions.

1. $f(x, y) = x + y + xy$
2. $h(x, y) = xe^y + y + 1$
3. $w = x^2 \tan(xy)$
4. $w = \frac{x-y}{x^2+y}$

Solution for (1), (2) and (3) in Exercise 26

1. $\frac{\partial f}{\partial x} = 1 + y$, $\frac{\partial f}{\partial y} = 1 + x$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$
2. $\frac{\partial h}{\partial x} = e^y$, $\frac{\partial h}{\partial y} = xe^y + 1$, $\frac{\partial^2 h}{\partial x^2} = 0$, $\frac{\partial^2 h}{\partial y^2} = xe^y$, $\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$
3. $\frac{\partial w}{\partial x} = 2x \tan(xy) + x^2 \sec^2(xy) \cdot y = 2x \tan(xy) + x^2 y \sec^2(xy)$,
   $\frac{\partial w}{\partial y} = x^2 \sec^2(xy) \cdot x = x^3 \sec^2(xy)$,
   $\frac{\partial^2 w}{\partial x^2} = 2 \tan(xy) + 2x \sec^2(xy) \cdot y + 2xy \sec^2(xy) + x^2 y(2 \sec(xy) \sec(xy) \tan(xy) \cdot y) =$
   $2 \tan(xy) + 4xy \sec^2(xy) + 2x^2 y^2 \sec^2(xy) \tan(xy)$,
   $\frac{\partial^2 w}{\partial y^2} = x^3(2 \sec(xy) \sec(xy) \tan(xy) \cdot x) = 2x^4 \sec^2(xy) \tan(xy)$
   $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2 \sec^2(xy) + x^3(2 \sec(xy) \sec(xy) \tan(xy) \cdot y) =$
   $3x^2 \sec^2(xy) + x^3 y \sec^2(xy) \tan(xy)$.

4. $$\frac{\partial w}{\partial x} = \frac{(x^2+y)-(x-y)(2x)}{(x^2+y)^2} = \frac{-x^2+2xy+y}{(x^2+y)^2},$$
$$\frac{\partial w}{\partial y} = \frac{(x^2+y)(-1)-(x-y)}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2},$$
$$\frac{\partial^2 w}{\partial x^2} = \frac{(x^2+y)^2(-2x+2y)-(-x^2+2xy+y)2(x^2+y)(2x)}{[(x^2+y)^2]^2} = \frac{2(x^3-3x^2y-3xy+y^2)}{(x^2+y)^3},$$
$$\frac{\partial^2 w}{\partial y^2} = \frac{(x^2+y)^2\cdot 0-(-x^2-x)2(x^2+y)\cdot 1}{[(x^2+y)^2]^2} = \frac{2x^2+2x}{(x^2+y)^3},$$
$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \frac{(x^2+y)^2(2x+1)-(-x^2+2xy+y)2(x^2+y)\cdot 1}{[(x^2+y)^2]^2} = \frac{2x^3+3x^2-2xy-y}{(x^2+y)^3}$$

Exercise 27.

1. *Verify that* $w_{xy} = w_{yx}$.
   (a) $w = e^x + x \ln y + y \ln x$
   (b) $w = x \sin y + y \sin x + xy$
2. *Which order of differentiation will calculate* f_{xy} *faster:* x *first or* y *first? try to answer without writing anything down.*
   (a) $f(x, y) = x \sin y + e^y$
   (b) $f(x, y) = y + x^2 y + 4y^3 - \ln(y^2 + 1)$
   (c) $f(x, y) = x \ln xy$

Solution for Exercise 27

1. (a) $\frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}$, $\frac{\partial w}{\partial y} = \frac{x}{y} + \ln x$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{1}{y} + \frac{1}{x}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$

   (b) $\frac{\partial w}{\partial x} = \sin y + y \cos x + y$, $\frac{\partial w}{\partial y} = x \cos y + \sin x + x$,
   $\frac{\partial^2 w}{\partial y \partial x} = \cos y + \cos x + 1$, and $\frac{\partial^2 w}{\partial x \partial y} = \cos y + \cos x + 1$

2. (a) x first

   (b) x first

   (c) y first

Exercise 28.

The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y? Try to answer without writing anything down.

1. $f(x,y) = y^2 + y(\sin x - x^4)$
2. $f(x,y) = xe^{y^2/2}$

Solution for Exercise 28

1. y first three times
2. x first twice

Using the Partial Derivative Definition

Exercise 29.

Use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

1. $f(x,y) = 1 - x + y - 3x^2y$, $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *at* $(1,2)$
2. $f(x,y) = \sqrt{2x + 3y - 1}$, $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *at* $(-2,3)$

Solution for Exercise 29

1. $f_x(1,2) = \lim\limits_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h} = \lim\limits_{h\to 0} \frac{\left[1-(1+h)+2-6(1+h)^2\right]-(2-6)}{h} =$
$\lim\limits_{h\to 0} \frac{-h-6(1+2h+h^2)+6}{h} \lim\limits_{h\to 0} \frac{-13h-6h^2}{h} = \lim\limits_{h\to 0}(-13-6h) = -13,$
$f_y(1,2) = \lim\limits_{h\to 0} \frac{f(1,2+h)-f(1,2)}{h} = \lim\limits_{h\to 0} \frac{[1-1+(2+h)-3(2+h)]-(2-6)}{h} =$
$\lim\limits_{h\to 0} \frac{(2-6-2h)-(2-6)}{h} = \lim\limits_{h\to 0}(-2) = -2$

2. $f_x(-2,3) = \lim\limits_{h\to 0} \frac{f(-2+h,3)-f(-2,3)}{h} = \lim\limits_{h\to 0} \frac{\sqrt{2(-2+h)+9-1}-\sqrt{-4+9-1}}{h}$
$= \lim\limits_{h\to 0} \frac{\sqrt{2h+4}-2}{h} = \lim\limits_{h\to 0}\left(\frac{\sqrt{2h+4}-2}{h}\frac{\sqrt{2h+4}+2}{\sqrt{2h+4}+2}\right) = \lim\limits_{h\to 0} \frac{2}{\sqrt{2h+4}+2} = \frac{1}{2},$
$f_y(-2,3) = \lim\limits_{h\to 0} \frac{f(-2,3+h)-f(-2,3)}{h} =$
$= \lim\limits_{h\to 0} \frac{\sqrt{-4+3(3+h)-1}-\sqrt{-4+9-1}}{h} = \lim\limits_{h\to 0} \frac{\sqrt{3h+4}-2}{h} =$
$\lim\limits_{h\to 0}\left(\frac{\sqrt{3h+4}-2}{h}\frac{\sqrt{3h+4}+2}{\sqrt{3h+4}+2}\right) = \lim\limits_{h\to 0} \frac{3}{\sqrt{2h+4}+2} = \frac{3}{4}$

Exercise 30.

1. *Let* $f(x, y) = 2x + 3y - 4$. *Find the slope of the line tangent to this surface at the point* $(2, -1)$ *and lying in the* a. *plane* $x = 2$ b. *plane* $y = -1$.
2. *Let* $f(x, y) = x^2 + y^3$. *Find the slope of the line tangent to this surface at the point* $(-1, 1)$ *and lying in the* a. *plane* $x = -1$ b. *plane* $y = 1$.
3. Three variables : *Let* $w = f(x, y, z)$ *be a function of three independent variables and write the formal definition of the partial derivative* $\partial f/\partial z$ *at* (x_0, y_0, z_0). *Use this definition to find* $\partial f/\partial z$ *at* $(1, 2, 3)$ *for* $f(x, y, z) = x^2yz^2$.
4. Three variables : *Let* $w = f(x, y, z)$ *be a function of three independent variables and write the formal definition of the partial derivative* $\partial f/\partial y$ *at* (x_0, y_0, z_0). *Use this definition to find* $\partial f/\partial y$ *at* $(-1, 0, 3)$ *for* $f(x, y, z) = -2xy^2 + yz^2$.

Solution for Exercise 30

1. (a) In the plane $x = 2 \Rightarrow f_y(x, y) = 3 \Rightarrow f_y(2, -1) = 3 \Rightarrow m = 3$
   (b) In the plane $y = -1 \Rightarrow f_x(x, y) = 2 \Rightarrow f_y(2, -1) = 2 \Rightarrow m = 2$
2. (a) In the plane
   $x = -1 \Rightarrow f_y(x, y) = 3y^2 \Rightarrow f_y(-1, 1) = 3(1)^2 = 3 \Rightarrow m = 3$
   (b) In the plane
   $y = 1 \Rightarrow f_x(x, y) = 2x \Rightarrow f_y(-1, 1) = 2(-1) = -2 \Rightarrow m = -2$
3. $f_z(x_0, y_0, z_0) = \lim\limits_{h \to 0} \frac{f(x_0,y_0,z_0+h) - f(x_0,y_0,z_0)}{h}$;
   $f_z(1, 2, 3) = \lim\limits_{h \to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h} = \lim\limits_{h \to 0} \frac{2(3+h)^2 - 2(9)}{h} =$
   $\lim\limits_{h \to 0} \frac{12h+2h^2}{h} = \lim\limits_{h \to 0} (12 + 2h) = 12$
4. $f_y(x_0, y_0, z_0) = \lim\limits_{h \to 0} \frac{f(x_0,y_0+h,z_0) - f(x_0,y_0,z_0)}{h}$;
   $f_y(-1, 0, 3) = \lim\limits_{h \to 0} \frac{f(-1,h,3) - f(-1,0,3)}{h} = \lim\limits_{h \to 0} \frac{(2h^2+5h) - 0}{h} =$
   $\lim\limits_{h \to 0} (2h + 9) = 9$

Exercise 31.

1. *Find the value of* $\partial z/\partial x$ *at the point* $(1, 1, 1)$ *if the equation*

$$xy + z^3x - 2yz = 0$$

defines z *as a function of the two independent variables* x *and* y *and the partial derivative exists.*

2. *Find the value of* $\partial x/\partial z$ *at the point* $(1, -1, -3)$ *if the equation*

$$xz + y \ln x - x^2 + 4 = 0$$

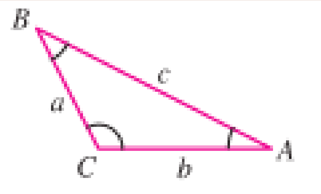
defines x *as a function of the two independent variables* y *and* z *and the partial derivative exists.*

Solution for Exercise 31

1. $y + \left(3z^2 \frac{\partial z}{\partial x}\right) x + z^3 - 2y \frac{\partial z}{\partial x} = 0 \Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial x} = -y - z^3 \Rightarrow$ at $(1, 1, 1)$ we have $(3 - 2) \frac{\partial z}{\partial x} = -1 - 1$ or $\frac{\partial z}{\partial x} = -2$
2. $\left(\frac{\partial x}{\partial z}\right) z + x + \left(\frac{y}{x}\right) \frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} = 0 \Rightarrow (z + \frac{y}{x} - 2x) \frac{\partial x}{\partial z} = -x \Rightarrow$ at $(1, -1, -3)$ we have $(-3 - 1 - 2) \frac{\partial x}{\partial z} = -1$ or $\frac{\partial x}{\partial z} = \frac{1}{6}$

Exercise 32.

The following exercises are about the triangle shown here.



1. *Express A implicitly as a function of a, b, and c and calculate $\partial A/\partial a$ and $\partial A/\partial b$.*
2. *Express a implicitly as a function of A, b, and B and calculate $\partial a/\partial A$ and $\partial a/\partial B$.*

Solution for Exercise 32

1. $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow 2a = (2bc\sin A)\frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc\sin A}$; also $0 = 2b - 2c\cos A + (2bc\sin A)\frac{\partial A}{\partial b} \Rightarrow 2c\cos A - 2b = (2bc\sin A)\frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} \Rightarrow = \frac{c\cos A - b}{bc\sin A}$
2. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A)\frac{\partial a}{\partial A} - a\cos A}{\sin A} = 0 \Rightarrow (\sin A)\frac{\partial a}{\partial x} - a\cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a\cos A}{\sin A}$; also $\left(\frac{1}{\sin A}\right)\frac{\partial a}{\partial B} = b(-\csc B\cot B) \Rightarrow \frac{\partial a}{\partial B} = -b\csc B\cot B\sin A$

Exercise 33.

1. *Express* v_x *in terms of* u *and* y *if the equations* $x = v \ln u$ *and* $y = u \ln v$ *define* u *and* v *as functions of the independent variables* x *and* y*, and if* v_x *exists.*
   (Hint : Differentiate both equations with respect to x *and solve for* v_x *by eliminating* u_x*.)*
2. Two dependent variables: *Find* $\partial x/\partial u$ *and* $\partial y/\partial u$ *if the equations* $u = x^2 - y^2$ *and* $v = x^2 - y$ *define* x *and* y *as functions of the independent variables* u *and* v*, and the partial derivatives exist. Then let* $s = x^2 + y^2$ *and find* $\partial s/\partial u$*.*

Solution for Exercise 33

1. Differentiating each equation implicitly gives $1 = v_x \ln u + \left(\frac{v}{u}\right) u_x$ and $0 = u_x \ln v + \left(\frac{u}{v}\right) v_x$ or

$$\left.\begin{aligned}(\ln u)v_x + \left(\frac{v}{u}\right) u_x &= 1\\ \left(\frac{u}{v}\right) v_x + (\ln v)u_x &= 0\end{aligned}\right\} \Rightarrow v_x = \frac{\begin{vmatrix}1 & \frac{v}{u}\\ 0 & \ln v\end{vmatrix}}{\begin{vmatrix}\ln u & \frac{v}{u}\\ \frac{u}{v} & \ln v\end{vmatrix}} = \frac{\ln v}{(\ln u)(\ln v)-1}$$

2. Differentiating each equation implicitly gives $1 = (2x)x_u - (2y)y_u$ and $0 = (2x)x_u - y_u$ or

$$\left.\begin{aligned}(2x)x_u - (2y)y_u &= 1\\ (2x)x_u - y_u &= 0\end{aligned}\right\} \Rightarrow x_u = \frac{\begin{vmatrix}1 & -2y\\ 0 & -1\end{vmatrix}}{\begin{vmatrix}2x & -2y\\ 2x & -1\end{vmatrix}} = \frac{-1}{(-2x+4xy)} = \frac{1}{2x-4xy}$$ and

$$y_u = \frac{\begin{vmatrix}2x & 1\\ 2x & 0\end{vmatrix}}{-2x+4xy} = \frac{-2x}{-2x+4xy} = \frac{2x}{2x-4xy} = \frac{1}{1-2y}$$; next $s = x^2 + y^2 \Rightarrow \frac{\partial s}{\partial u} = 2x\frac{\partial x}{\partial u} + 2y\frac{\partial y}{\partial u}$

$$2x\left(\frac{1}{2x-4xy}\right) + 2y\left(\frac{1}{1-2y}\right) = \frac{1}{1-2y} + \frac{2y}{1-2y} = \frac{1+2y}{1-2y}$$

Exercise 34.

1. *Let* $f(x,y) = \begin{cases} y^3, & y \geq 0 \\ -y^2, & y < 0. \end{cases}$
   Find f_x, f_y, f_{xy}*, and* f_{yx}*, and state the domain for each partial derivative.*
2. *Let* $f(x,y) = \begin{cases} \sqrt{x}, & x \geq 0 \\ x^2, & x < 0. \end{cases}$
   Find f_x, f_y, f_{xy}*, and* f_{yx}*, and state the domain for each partial derivative.*

Solution for Exercise 34

1. $f_x(x,y) = \begin{cases} 0 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases} \Rightarrow f_x(x,y) = 0$ for all points (x,y); at $y = 0$,
$f_y(x,0) = \lim\limits_{h\to 0} \frac{f(x,0+h)-f(x,0)}{h} = \lim\limits_{h\to 0} \frac{f(x,h)-0}{h} = \lim\limits_{h\to 0} \frac{f(x,h)}{h} = 0$ because
$\lim\limits_{h\to 0^-} \frac{f(x,h)}{h} = \lim\limits_{h\to 0^-} \frac{h^3}{h} = 0$ and
$\lim\limits_{h\to 0^+} \frac{f(x,h)}{h} = \lim\limits_{h\to 0^+} \frac{h^2}{h} = 0 \Rightarrow f_y(x,y) = \begin{cases} 3y^2 & \text{if } y \geq 0 \\ -2y & \text{if } y < 0 \end{cases}$; $f_{yx}(x,y) = f_{xy}(x,y) = 0$ for all points (x,y).
2. At $x = 0$, $f_x(0,y) = \lim\limits_{h\to 0} \frac{f(0+h,y)-f(0,y)}{h} = \lim\limits_{h\to 0} \frac{f(h,y)-0}{h} = \lim\limits_{h\to 0} \frac{f(h,y)}{h}$ which does not exist because $\lim\limits_{h\to 0^-} \frac{f(h,y)}{h} = \lim\limits_{h\to 0^-} \frac{h^2}{h} = 0$ and
$\lim\limits_{h\to 0^+} \frac{f(h,y)}{h} = \lim\limits_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim\limits_{h\to 0^+} \frac{1}{\sqrt{h}} = +\infty \Rightarrow f_x(x,y) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0 \\ 2x & \text{if } x < 0 \end{cases}$;
$f_y(x,y) = \begin{cases} 0 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \Rightarrow f_y(x,y) = 0$ for all points (x,y); $f_{yx}(x,y) = 0$ for all points (x,y), while $f_{xy}(x,y) = 0$ for all points (x,y) such that $x \neq 0$.

Exercise 35.

Show that each function satisfies a Laplace equation.

1. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$
2. $f(x, y) = \tan^{-1} \frac{x}{y}$
3. $f(x, y, z) = e^{3x+4y} \cos 5z$

Solution for Exercise 35

1. $\frac{\partial f}{\partial x} = -6xz$, $\frac{\partial f}{\partial y} = -6yz$, $\frac{\partial f}{\partial z} = 6z^2 - 3(x^2 + y^2)$, $\frac{\partial^2 f}{\partial x^2} = -6z$,
   $\frac{\partial^2 f}{\partial y^2} = -6z$, $\frac{\partial^2 f}{\partial z^2} = 12z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$
2. $\frac{\partial f}{\partial x} = \frac{1/y}{1+\left(\frac{x}{y}\right)^2} = \frac{y}{y^2+x^2}$, $\frac{\partial f}{\partial y} = \frac{-x/y^2}{1+\left(\frac{x}{y}\right)^2} = \frac{-x}{y^2+x^2}$,
   $\frac{\partial^2 f}{\partial x^2} = \frac{(y^2+x^2)\cdot 0 - y\cdot 2x}{(y^2+x^2)^2} = \frac{-2xy}{(y^2+x^2)^2}$, $\frac{\partial^2 f}{\partial y^2} = \frac{(y^2+x^2)\cdot 0-(-x)\cdot 2y}{(y^2+x^2)^2} = \frac{2xy}{(y^2+x^2)^2} \Rightarrow$
   $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(y^2+x^2)^2} + \frac{2xy}{(y^2+x^2)^2} = 0$
3. $\frac{\partial f}{\partial x} = 3e^{3x+4y}\cos 5z$, $\frac{\partial f}{\partial y} = 4e^{3x+4y}\cos 5z$, $\frac{\partial f}{\partial z} = -5e^{3x+4y}\sin 5z$;
   $\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y}\cos 5z$, $\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y}\cos 5z$,
   $\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y}\cos 5z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} =$
   $9e^{3x+4y}\cos 5z + 16e^{3x+4y}\cos 5z - 25e^{3x+4y}\cos 5z = 0$

Exercise 36.

Show that the functions are all solutions of the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.

1. $w = \cos(2x + 2ct)$
2. $w = \ln(2x + 2ct)$
3. $w = f(u)$, *where f is a differentiable function of u, and $u = a(x + ct)$, where a is a constant.*
4. *Does a function $f(x, y)$ with continuous first partial derivatives throughout an open region R have to be continuous on R? Give reasons for your answer.*
5. *If a function $f(x, y)$ has continuous second partial derivatives throughout an open region R, must the first-order partial derivatives of f be continuous on R? Give reasons for your answer.*

Solution for (1), (2) and (3) in Exercise 36

1. $\frac{\partial w}{\partial x} = -2\sin(2x+2ct)$, $\frac{\partial w}{\partial t} = -2c\sin(2x+2ct)$;
   $\frac{\partial^2 w}{\partial x^2} = -4\cos(2x+2ct)$,
   $\frac{\partial^2 w}{\partial t^2} = -4c^2\cos(2x+2ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-4\cos(2x+2ct)] = c^2\frac{\partial^2 w}{\partial x^2}$
2. $\frac{\partial w}{\partial x} = \frac{1}{x+ct}$, $\frac{\partial w}{\partial t} = \frac{c}{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x+ct)^2}$,
   $\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[\frac{-1}{(x+ct)^2}\right] = c^2\frac{\partial^2 w}{\partial x^2}$
3. $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(ac) \Rightarrow \frac{\partial^2 w}{\partial t^2} = (ac)\frac{\partial^2 f}{\partial u^2}(ac) = a^2c^2\frac{\partial^2 f}{\partial u^2}$;
   $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}\cdot a \Rightarrow \frac{\partial^2 w}{\partial x^2} = \left(a\frac{\partial^2 f}{\partial u^2}\right)\cdot a = a^2\frac{\partial^2 f}{\partial u^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} =$
   $a^2c^2\frac{\partial^2 f}{\partial u^2} = c^2\left(a^2\frac{\partial^2 f}{\partial u^2}\right) = c^2\frac{\partial^2 w}{\partial x^2}$

Solution for (4) and (5) in Exercise 36

1. If the first partial derivatives are continuous throughout an open region R, then by Theorem 3 in this section of the text, $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$, where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Then as $(x,y) \to (x_0,y_0)$, $\Delta x \to 0$ and $\Delta y \to 0 \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \Rightarrow f$ is continuous at every point (x_0,y_0) in R.
2. Yes, since f_{xx}, f_{yy}, f_{xy}, and f_{yx} are all continuous on R, use the same reasoning as in Exercise 76 with $f_x(x,y) = f_x(x_0,y_0) + f_{xx}(x_0,y_0)\Delta x + f_{xy}(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ and $f_y(x,y) = f_y(x_0,y_0) + f_{yx}(x_0,y_0)\Delta x + f_{yy}(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$. Then $\lim_{(x,y)\to(x_0,y_0)} f_x(x,y) = f_x(x_0,y_0)$ and $\lim_{(x,y)\to(x_0,y_0)} f_y(x,y) = f_y(x_0,y_0)$.

Exercise 37.

An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

Show that $u(x, t) = \sin(\alpha x) \cdot e^{-\beta t}$ *satisfies the heat equation for constants* α *and* β.

Solution for Exercise 37

To find α and β so that $u_t = u_{xx} \Rightarrow u_t = -\beta \sin(\alpha x) e^{-\beta t}$ and $u_x = \alpha \cos(\alpha x) e^{-\beta t} \Rightarrow u_{xx} = -\alpha^2 \sin(\alpha x) e^{-\beta t}$; then $u_t = u_{xx} \Rightarrow -\beta \sin(\alpha x) e^{-\beta t} = -a^2 \sin(ax) e^{-\beta t}$, thus $u_t = u_{xx}$ only if $\beta = \alpha^2$

Exercise 38.

1. *Let* $f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$
   Show that $f_x(0, 0)$ *and* $f_y(0, 0)$ *exist, but* f *is not differentiable at* $(0, 0)$.
   (Hint:Show that f *is not continuous at* $(0, 0)$*.)*
2. *Let* $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \textit{otherwise}. \end{cases}$
   Show that $f_x(0, 0)$ *and* $f_y(0, 0)$ *exist, but* f *is not differentiable at* $(0, 0)$.

Solution for Exercise 38

1. $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h0^2}{h^2+0^4}-0}{h} = \lim_{h\to 0} \frac{0}{h} = 0;$

   $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} = \frac{\frac{0h^2}{0^2+h^4}-0}{h} = \lim_{h\to 0} \frac{0}{h} = 0;$

   $\lim_{\substack{(x,y)\to(0,0)\\ \text{along} x=ky^2}} f(x,y) = \lim_{y\to 0} \frac{(ky^2)y^2}{(ky^2)^2+y^4} = \lim_{y\to 0} \frac{ky^4}{k^2y^4+y^4} = \lim_{y\to 0} \frac{k}{k^2+1} = \frac{k}{k^2+1} \Rightarrow$ different limits for different values of $k \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f(x,y)$ is not continuous at $(0,0) \Rightarrow$ by Theorem 18 (Differentiability Implies Continuity), $f(x,y)$ is not differentiable at $(0,0)$.

2. $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)-1}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0;$

   $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)-1}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0;$

   $\lim_{\substack{(x,y)\to(0,0)\\ \text{along} y=x^2}} f(x,y) = \lim_{y\to 0} 0 = 0$ but $\lim_{\substack{(x,y)\to(0,0)\\ \text{along} y=1.5x^2}} f(x,y) = \lim_{y\to 0} 1 = 1 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f(x,y)$ is not continuous at $(0,0) \Rightarrow$ by Theorem 18 (Differentiability Implies Continuity), $f(x,y)$ is not differentiable at $(0,0)$.

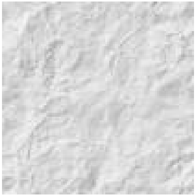
Extra Notes on Differentiability

An informal definition of differentiability of a function $f(x, y)$ at a point (a, b) is that the function $f(x, y)$ is well-approximated by a linear function $L(x, y)$ near (a, b). The function $L(x, y)$ is called the **local linearization** of $f(x, y)$ near (a, b). We now focus on the precise meaning of the phrase "well-approximated."

By looking at examples, we shall see that local linearity requires the existence of partial derivatives, but they do not tell the whole story.

In particular, existence of partial derivatives at a point is not sufficient to guarantee local linearity at that point.

As an illustration, take a sheet of paper, crumple it into a ball and smooth it out again. Wherever there is a crease it would be difficult to approximate the surface by a plane — these are points of nondifferentiability of the function giving the height of the paper above the floor. Yet the sheet of paper models a graph which is continuous — there are no breaks.

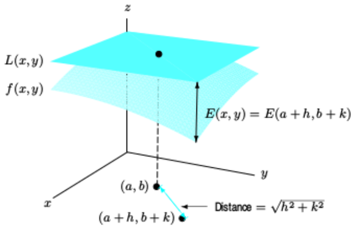


As in the case of one-variable calculus, continuity does not imply differentiability. But differentiability does require continuity: there cannot be linear approximations to a surface at points where there are abrupt changes in height.

Extra Notes on Differentiability

For a function of two variables, as for a function of one variable, we define **differentiability at a point in terms of the error in a linear approximation** as we move from the point to a nearby point.

If the point is (a, b) and a nearby point is $(a + h, b + k)$, the distance between them is $\sqrt{h^2 + k^2}$.



Definition 39.

A function $f(x,y)$ *is* differentiable at the point (a,b) *if there is a linear function* $L(x,y) = f(a,b) + m(x-a) + n(y-b)$ *such that if the error* $E(x,y)$ *is defined by*

$$f(x,y) = L(x,y) + E(x,y),$$

and if $h = x - a, \; k = y - b$, *then the* relative error $\frac{E(a+h,b+k)}{\sqrt{h^2+k^2}}$ *satisfies*

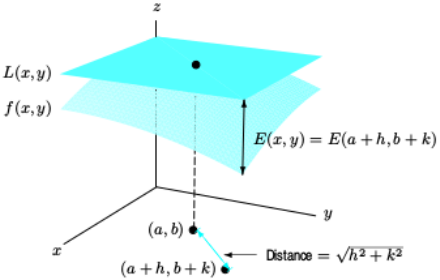
$$\lim_{(h,k)\to(0,0)} \frac{E(a+h,b+k)}{\sqrt{h^2+k^2}} = 0.$$

The function f *is* differentiable *if it is differentiable at each point of its domain. The function* $L(x,y)$ *is called the* local linearization *of* $f(x,y)$ *near* (a,b).

Extra Notes on Differentiability

A function $f(x, y)$ is **differentiable at the point** (a, b) if there is a linear function $L(x, y) = f(a, b) + m(x - a) + n(y - b)$ such that if the error $E(x, y)$ is defined by $f(x, y) = L(x, y) + E(x, y)$, and if $h = x - a,\ k = y - b$, then the **relative error** $\frac{E(a+h, b+k)}{\sqrt{h^2+k^2}}$ satisfies

$$\lim_{(h,k)\to(0,0)} \frac{E(a+h, b+k)}{\sqrt{h^2+k^2}} = 0.$$



In the next example, we show that this definition of differentiability is consistent with our previous notion — that is, that $m = f_x$ and $n = f_y$ and that the graph of $L(x, y)$ is the tangent plane.

Theorem 40.

Show that if f is a differentiable function with local linearization

$$L(x, y) = f(a, b) + m(x - a) + n(y - b),$$

then $m = f_x(a, b)$ *and* $n = f_y(a, b)$.

Proof : Since f is differentiable, we know that the relative error in $L(x, y)$ tends to 0 as we get close to (a, b).

Solution (contd...)

Suppose $h > 0$ and $k = 0$. Then we know that

$$
\begin{aligned}
0 = \lim_{h \to 0} \frac{E(a+h, b)}{\sqrt{h^2 + k^2}} &= \lim_{h \to 0} \frac{E(a+h, b)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h, b) - L(a+h, b)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h, b) - f(a, b) - mh}{h} \\
&= \lim_{h \to 0} \left(\frac{f(a+h, b) - f(a, b)}{h} \right) - m \\
&= f_x(a, b) - m.
\end{aligned}
$$

A similar result holds if $h < 0$, so we have $m = f_x(a, b)$. The result $n = f_y(a, b)$ is found in a similar manner.

Extra Notes on Differentiability

The previous result says that if a function is differentiable at a point, it has partial derivatives there.

Therefore, if any of the partial derivatives fail to exist, then the function cannot be differentiable.

This is what happens in the following example of a cone.

Example 41.

Consider the function

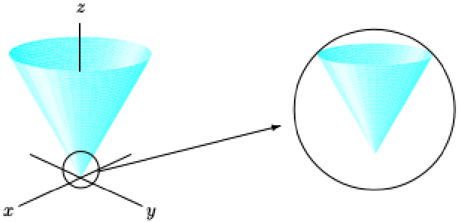
$$f(x, y) = \sqrt{x^2 + y^2}.$$

Is f differentiable at the origin?

Extra Notes on Differentiability

The function $f(x, y) = \sqrt{x^2 + y^2}$ is not locally linear at $(0, 0)$.

Zooming in around $(0, 0)$ does not make the graph look like a plane.



If we zoom in on the graph of the function $f(x, y) = \sqrt{x^2 + y^2}$ at the origin, as shown in the figure, the sharp point remains; the graph never flattens out to look like a plane. Near its vertex, the graph does not look like it is well approximated (in any reasonable sense) by any plane.

Solution

Judging from the graph of f, we would not expect f to be differentiable at $(0,0)$. Let us check this by trying to compute the partial derivatives of f at $(0,0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

Since $|h|/h = \pm 1$, depending on whether h approaches 0 from the left or right, this limit does not exist and so neither does the partial derivative $f_x(0,0)$.

Thus, f cannot be differentiable at the origin. If it were, both of the partial derivatives, $f_x(0,0)$ and $f_y(0,0)$, would exist.

Alternate Solution

Alternatively, we could show directly that there is no linear approximation near $(0, 0)$ that satisfies the small relative error criterion for differentiability.

Any plane passing through the point $(0, 0, 0)$ has the form

$$L(x, y) = mx + ny$$

for some constants m and n.

If $E(x, y) = f(x, y) - L(x, y)$, then

$$E(x, y) = \sqrt{x^2 + y^2} - mx - ny.$$

Then for f to be differentiable at the origin, we would need to show that

$$\lim_{(h,k)\to(0,0)} \frac{\sqrt{h^2 + k^2} - mh - nk}{\sqrt{h^2 + k^2}} = 0.$$

Alternate Solution (contd...)

Taking $k = 0$ gives

$$\lim_{h\to 0} \frac{|h| - mh}{|h|} = 1 - m \lim_{h\to 0} \frac{h}{|h|}.$$

This limit exists only if $m = 0$ for the same reason as before. But then the value of the limit is 1 and not 0 as required. Thus, we again conclude f is not differentiable.

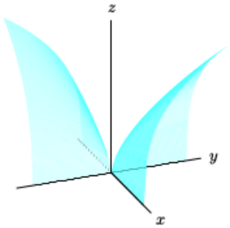
In the above example the partial derivatives f_x and f_y did not exist at the origin and this was sufficient to establish nondifferentiability there. We might expect that if both partial derivatives do exist, then f is differentiable. But the next example shows that this not necessarily true: the existence of both partial derivatives at a point is not sufficient to guarantee differentiability.

Example 42.

Consider the function

$$f(x,y) = x^{1/3}y^{1/3}.$$

Show that the partial derivatives $f_x(0,0)$ *and* $f_y(0,0)$ *exist, but that* f *is not differentiable at* $(0,0)$.



Graph of $z = x^{1/3}y^{1/3}$ for $z \geq 0$

Solution

We have $f(0,0) = 0$ and we compute the partial derivatives using the definition:

$$\lim_{h\to 0} f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} 0 - 0 = 0,$$

and similarly

$$f_y(0,0) = 0.$$

So, if there did exist a linear approximation near the origin, it would have to be $L(x,y) = 0$. But we can show that this choice of $L(x,y)$ does not result in the small relative error that is required for differentiability.

In fact, since $E(x,y) = f(x,y) - L(x,y) = f(x,y)$, we need to look at the limit

$$\lim_{(h,k)\to(0,0)} \frac{h^{1/3}\, k^{1/3}}{\sqrt{h^2 + k^2}}.$$

Solution (contd...)

If this limit exists, we get the same value no matter how h and k approach 0. Suppose we take $k = h > 0$. Then the limit becomes

$$\lim_{h \to 0} \frac{h^{1/3} h^{1/3}}{\sqrt{h^2 + k^2}} = \lim_{h \to 0} \frac{h^{2/3}}{h\sqrt{2}} = \lim_{h \to 0} \frac{1}{h^{1/3}\sqrt{2}}.$$

But this limit does not exist, since small values for h will make the fraction arbitrarily large. So the only possible candidate for a linear approximation at the origin does not have a sufficiently small relative error.

Thus, this function is not differentiable at the origin, even though the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist. The figure confirms that near the origin the graph of $z = f(x, y)$ is not well approximated by any plane.

In the above example, the function f was continuous at the point where it was not differentiable.

Summary

- If a function is differentiable at a point, then both partial derivatives exist there.
- Having both partial derivatives at a point does not guarantee that a function is differentiable there.

We know that differentiable functions of one variable are continuous. Similarly, it can be shown that if a function of two variables is differentiable at a point, then the function is continuous there. The following example shows that even if the partial derivatives of a function exist at a point, the function is not necessarily continuous at that point if it is not differentiable there.

Example 43.

Suppose that f is the function of two variables defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

One can show that $f(x,y)$ is not continuous at the origin. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist. Could f be differentiable at $(0,0)$?

Solution

From the definition of the partial derivative we see that

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \left(\frac{1}{h} . \frac{0}{h^2 + 0^2} \right) = \lim_{h \to 0} \frac{0}{h} = 0,$$

and similarly $f_y(0,0) = 0$.

So, the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist.

However, f cannot be differentiable at the origin since it is not continuous there.

Summary

- If a function is differentiable at a point, then it is continuous there.
- Having both partial derivatives at a point does not guarantee that a function is continuous there.

How Do We Know If a Function Is Differentiable?

Question : Can we use partial derivatives to tell us if a function is differentiable?

As we have seen from the last two examples, it is not enough that the partial derivatives exist.

However, the following condition does guarantee differentiability:

Theorem 44 (Condition for Differentiability).

If the partial derivatives, f_x and f_y, of a function f exist and are continuous on a small disk centered at the point (a, b), then f is differentiable at (a, b).

Condition for Differentiability

We will not prove this fact, although it provides a criterion for differentiability which is often simpler to use than the definition.

It turns out that the requirement of continuous partial derivatives is more stringent than that of differentiability, so there exist differentiable functions which do not have continuous partial derivatives.

However, most functions we encounter will have continuous partial derivatives.

The class of functions with continuous partial derivatives is given the name C^1.

Example 45.

Show that the function $f(x, y) = \ln(x^2 + y^2)$ *is differentiable everywhere in its domain.*

Solution : The domain of f is all of 2-space except for the origin. We shall show that f has continuous partial derivatives everywhere in its domain (that is, the function f is in C^1). The partial derivatives are

$$f_x = \frac{2x}{x^2 + y^2} \quad \text{and} \quad f_y = \frac{2y}{x^2 + y^2}.$$

Since each of f_x and f_y is the quotient of continuous functions, the partial derivatives are continuous everywhere except the origin (where the denominators are zero). Thus, f is differentiable everywhere in its domain.

Exercise 46.

The three-dimensional Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ *is satisfied by steady-state temperature distributions* $T = f(x, y, z)$ *in space, by gravitational potentials, and by electrostatic potentials. The* two-dimensional Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, *obtained by dropping the* $\partial^2 f/\partial z^2$ *term from the previous equation, describes potentials and steady-state temperature distributions in a plane (see the accompanying figure). The plane (a) may be treated as a thin slice of the solid (b) perpendicular to the z-axis.*

(a). $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

(b). $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

Exercise 47.

If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where w *is the wave height,* x *is the distance variable,* t *is the time variable, and* c *is the velocity with which the waves are propagated. In our example,* x *is the distance across the ocean's surface, but in other applications,* x *might be the distance along a vibrating string, distance through air (sound waves), or distance through space (light waves). The number* c *varies with the medium and type of wave.*

Summary

Most functions built up from elementary functions have continuous partial derivatives, except perhaps at a few obvious points.

Thus, in practice, we can often identify functions as being C^1 without explicitly computing the partial derivatives.

Question 48.

Given $f(x,y) = \begin{cases} x^2 y \sin\frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$. *Then*

(A) *f is differentiable at $(0,0)$.*

(B) *f is not differentiable at $(0,0)$.*

(C) *$f_x(a,0)$ exists, but $f_y(a,0)$ does not exist, where $a \neq 0$.*

(D) *Both $f_x(a,0)$ and $f_y(a,0)$ exist, where $a \neq 0$.*

(E) *None of the other options.*

The correct answers are
f is differentiable at $(0,0)$; $f_x(a,0)$ exists, but $f_y(a,0)$ does not exist, where $a \neq 0$.

Question 49.

Given $f(x,y) = \begin{cases} xy^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. *Then*

(A) f *is differentiable at* $(0,0)$.

(B) f *is not differentiable at* $(0,0)$.

(C) $f_x(0,b)$ *does not exist, but* $f_y(0,b)$ *exist, where* $b \neq 0$.

(D) *Both* $f_x(0,b)$ *and* $f_y(0,b)$ *do not exist, where* $b \neq 0$.

(E) *None of the other options.*

The correct answers are
f is differentiable at $(0,0)$; $f_x(0,b)$ does not exist, but $f_y(0,b)$ exist, where $b \neq 0$.

Question 50.

Given $f(x,y) = \begin{cases} x^2 \sin \frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$. *Then*

(A) *f is differentiable at $(0,0)$.*

(B) *f is not differentiable at $(0,0)$.*

(C) *$f_x(a,0)$ exists, but $f_y(a,0)$ does not exist, where $a \neq 0$.*

(D) *Both $f_x(a,0)$ and $f_y(a,0)$ exist, where $a \neq 0$.*

(E) *None of the other options.*

The correct answers are
f is differentiable at $(0,0)$; $f_x(a,0)$ exists, but $f_y(a,0)$ does not exist, where $a \neq 0$.

Question 51.

Given $f(x,y) = \begin{cases} y^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. *Then*

(A) f *is differentiable at* $(0,0)$.

(B) f *is not differentiable at* $(0,0)$.

(C) $f_x(0,b)$ *does not exist, but* $f_y(0,b)$ *exist, where* $b \neq 0$.

(D) *Both* $f_x(0,b)$ *and* $f_y(0,b)$ *do not exist, where* $b \neq 0$.

(E) *None of the other options.*

The correct answers are
f is differentiable at $(0,0)$; $f_x(0,b)$ does not exist, but $f_y(0,b)$ exist, where $b \neq 0$.

Old Questions

Q.3. Which of the following is/are true for $f(x, y) = |x|$?

(a) $f_x(1, 0) = 1$

(b) $f_y(1, 0) = 0$

(c) $f_x(1, 0)$ does not exist

(d) $f_y(1, 0)$ does not exist

Solution: (a) and (b)

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